General Correction Mechanisms for Weakening or Reinforcing Belief Functions

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Abstract - ***The discounting operation is a well known operation on belief functions, which has proved to be useful in many applications. However, the discounting operation only allows one to weaken a source, whereas it is sometimes useful to strengthen it when it is deemed to be too cautious. For that purpose, the de-discounting operation was introduced as the inverse operation of the discounting operation by Denœux and Smets. From another point of view, Zhu and Basir introduced an extension of the classical discounting operation by allowing the discount rate to be out of the range [0,1]. This operation performs a discounting or a de-discounting of a belief function. A new interpretation of this scheme is presented in this paper. A more general form of reinforcement process, as well as a parameterized family of transformations encompassing all previous schemes, are also introduced.***

Keywords: Dempster-Shafer theory, Evidence theory, discounting, de-discounting.

1 Introduction

In 1976, in his seminal book [7], Shafer introduced the discounting operation, allowing one to take into account the reliability of a source of information, and transforming a belief function into a weakened or less informative one. Nowadays, this discounting operation is an important tool of the Dempster-Shafer theory of belief functions. It is used in many applications, particularly those dealing with information fusion. The discounting operation is controled by a constant $\alpha \in [0, 1]$ called the *discount rate*. Smets [1] justified this operation [9] in the Transferable Belief Model (TBM) framework [10, 12], using a simple model of reliability: if the source is reliable, then information provided by this source is totally accepted; otherwise, the source is not reliable, and the information coming from this source is neglected. The quantity $1 - \alpha$ is interpreted as a degree of belief in the reliability of the source.

The discounting operation allows one to "tune down" a source. However, when a source is too cautious, it could be beneficial to give more weight to the information it provides, i.e., to reinforce the source's beliefs.

In [2], Denœux and Smets have introduced the notion of *de-discounting* of a belief function, as the inverse of the discounting process: instead of uniformly transferring a fraction of the mass from each focal set to the universe Ω, a fraction of the mass of the universe can be uniformly dispatched to each focal element, as long as $m(\Omega) < 1$. Thus, the de-discounting allows one to transform a belief function into a strengthened, reinforced, or more informative one.

It would be convenient to have a single tool for manipulating belief functions including these two possibilities: weakening or reinforcing a belief function, what is called here a *general correction mechanism.* In [14], Zhu and Basir recognized the necessity to extend the discounting process, in order to augment or discount belief functions in particular. They achieved this goal by retaining the discounting equation, and allowing the discount rate α to be in the range $[\frac{-m_S^\Omega(\Omega)}{1-m_S^\Omega(\Omega)}, 1]$. If $\alpha \in [0, 1]$, ${}^\alpha m$ is a discounting of m_S^Ω; if $\alpha \in [\frac{-m_S^\Omega(\Omega)}{1-m_S^\Omega(\Omega)}, 0]$, ${}^\alpha m$ is a de-discounting of m_S^Ω, as shown below; if $\alpha = 0$, ${}^\alpha m$ remains equals to m_S^Ω. Although this extended scheme was shown to be useful in a real-world application [14], it lacks a proper justification, since it is no longer possible to interpret $1 - \alpha$ as a degree of belief in the reliability of the source. Such an interperation is proposed below.

In this paper, the Transferable Belief Model (TBM) framework is accepted as a general model of uncertainty. The TBM is a subjectivist, non probabilistic interpretation of the Dempster-Shafer theory of belief functions [7, 11]. However, our approach is compatible with other interpretations.

We first recall some basic concepts on the TBM, and introduce some definitions as well as the main tools used in this paper. Then, a general reinforcement process is introduced, of which the de-discounting process is a particular case. Next, general corrections allowing discounting and de-discounting operations are tackled. Zhu and Basir's scheme is shown to be the result of the discounting process applied to the maximal de-discounted belief function. A property on the mass given to the universe is presented, and a new general transformation with more degrees of freedom is eventually introduced and justified.

2 Basic Concepts

2.1 Main definitions

Let x be a variable taking values in a finite set $\Omega = \{\omega_1, \ldots, \omega_K\}$, called *the frame of discernment*. The knowledge held by a rational agent Ag (the *belief holder*) regarding the actual value ω_0 taken by x, at a time t, given an evidential corpus EC, can be quantified by a basic belief assignment (BBA) $m^\Omega_{Ag,t}[EC]$, defined as a function from 2^Ω to $[0, 1]$ verifying:

$$\sum_{A \subseteq \Omega} m^\Omega_{Ag,t}[EC](A) = 1 \ . \tag{1}$$

When there is no ambiguity, the full notation $m^\Omega_{Ag,t}[EC]$ will be simplified to m^Ω_{Ag}, m^Ω, or even m. Note that in the TBM, BBAs are not required to be normalized, i.e., we may have $m(\emptyset) > 0$. The interpretation of $m(\emptyset)$ is discussed in [8], and more recently in an interesting review [13].

A BBA m is in one-to-one correspondence with a *belief function* (**BF**) bel : $2^\Omega \to [0, 1]$ and a *plausibility function* pl : $2^\Omega \to [0, 1]$ defined, respectively, as:

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \tag{2}$$

and

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \quad \forall A \subseteq \Omega. \tag{3}$$

Definition 1 (Focal elements) *Subsets A of Ω such that $m(A) > 0$ are called* focal elements *of m.*

Definition 2 (Categorical BF) *A* categorical belief function *focused on a subset B of Ω is defined, such that its related BBA m^Ω_B satisfies:*

$$m^\Omega_B(A) = \begin{cases} 1 & \text{if } A = B \ , \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Note that, m_B denotes a categorical belief function focused on B, only if B is a subset of Ω; otherwise m_{Ag} is a BBA provided by a belief holder Ag. In this paper, belief holders can be noted by Ag (a rational agent) or S (a source of information).

Definition 3 (Vacuous BF) *The* vacuous belief function *(VBF) is a categorical belief function focused on Ω. It represents total ignorance.*

Definition 4 (Bayesian BF) *A* Bayesian belief function *on Ω is a belief function on Ω such that its focal elements are singletons of Ω.*

Definition 5 (Commitment) *A belief function with related plausibility pl_2 is said to be* not more committed *(and* less committed *if there is at least one strict inegalities) than a belief function with related plausibility pl_1 if and only if:*

$$\text{pl}_1(A) \leq \text{pl}_2(A) \ \ \forall A \subseteq \Omega. \tag{5}$$

Definition 6 (Minimal Commitment Principle) *Among a set of belief functions in agreement with available information, the* principle of minimal commitment *consists in choosing the least committed belief function.*

This principle reflects a form of *scepticism* and the desire to precisely model the available information without introducing any unjustified pieces of information [3, 5, 9]. It is at the origin of the *vacuous* and *ballooning extensions* recalled below.

Combining two BBAs Two distinct and reliable BBAs m_1 and m_2 can be combined using the *conjunctive rule of combination* (CRC) defined by:

$$m_1 \textcircled{\cap} m_2(A) = \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega.$$

Marginalization and Vacuous Extension A BBA defined on a product space $\Omega \times \Theta$ may be marginalized on Ω, by transferring each mass $m^{\Omega \times \Theta}(B)$ for $B \subseteq \Omega \times \Theta$ to its projection on Ω:

$$m^{\Omega \times \Theta \downarrow \Omega}(A) = \sum_{\{B \subseteq \Omega \times \Theta \ | \ \text{Proj}(B \downarrow \Omega) = A\}} m^{\Omega \times \Theta}(B) \ , \quad \forall A \subseteq \Omega \tag{6}$$

where $\text{Proj}(B \downarrow \Omega)$ denotes the projection of B onto Ω.

It is usually not possible to retrieve the original BBA $m^{\Omega \times \Theta}$ from its marginal $m^{\Omega \times \Theta \downarrow \Omega}$ on Ω. However, the least committed BBA such that its projection on Ω is $m^{\Omega \times \Theta \downarrow \Omega}$ may be computed. This defines the *vacuous extension* of m^Ω in the product space $\Omega \times \Theta$ [9], noted $m^{\Omega \uparrow \Omega \times \Theta}$, given by:

$$m^{\Omega \uparrow \Omega \times \Theta}(B) = \begin{cases} m^\Omega(A) & \text{if } B = A \times \Theta, \ A \subseteq \Omega \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Conditioning and Ballooning Extension Conditional beliefs represent knowledge which is valid provided that an hypothesis is satisfied. Let m be a BBA, $B \subseteq \Omega$ an hypothesis; the *conditional belief function* $m[B]$ is:

$$m[B] = m \textcircled{\cap} m_B. \tag{8}$$

If $m^{\Omega \times \Theta}$ is defined on the product space $\Omega \times \Theta$, and θ is a subset of Θ, the conditional BBA $m^\Omega[\theta]$ is defined by combining $m^{\Omega\times\Theta}$ with $m_\theta^{\Theta\uparrow\Omega\times\Theta}$, and marginalizing the result on Ω:

$$m^\Omega[\theta] = \left(m^{\Omega\times\Theta} \circledcirc m_\theta^{\Theta\uparrow\Omega\times\Theta}\right)^{\downarrow\Omega} \tag{9}$$

Assume now that $m^\Omega[\theta]$ represents the agent's beliefs on Ω conditionally on θ, i.e., in a context where θ holds. There are usually many BBAs on $\Omega\times\Theta$, whose conditioning on θ yields $m^\Omega[\theta]$. Among these, the least committed one is the *ballooning extension* [9] defined by:

$$m^\Omega[\theta]^{\Uparrow\Omega\times\Theta}(A\times\theta\cup\Omega\times\overline{\theta}) = m^\Omega[\theta](A), \quad \forall A\subseteq\Omega. \tag{10}$$

$m^\Omega[\theta]^{\Uparrow\Omega\times\Theta}$ is also called a *deconditioning* of $m^\Omega[\theta]$ on $\Omega\times\Theta$.

2.2 Discounting

Let us assume that an agent Ag receives a BBA m_S^Ω from a source S, describing the source's beliefs regarding the actual value ω_0. Moreover, Ag has some knowledge about the *reliability* of S, quantified by a BBA $m_{Ag}^{\mathcal{R}}$ on the space $\mathcal{R}=\{R,NR\}$, where R stands for *"the source is reliable"*, and NR for *"the source is not reliable"* [9]. Let us assume that $m_{Ag}^{\mathcal{R}}$ has the following form:

$$\left\{\begin{array}{lcl} m_{Ag}^{\mathcal{R}}(\{R\}) & = & 1-\alpha \\ m_{Ag}^{\mathcal{R}}(\mathcal{R}) & = & \alpha, \end{array}\right. \tag{11}$$

for some $\alpha\in[0,1]$.

If S is reliable, the information provided by S becomes Ag's knowledge:

$$m_{Ag}^\Omega[R] = m_S^\Omega, \tag{12}$$

where the notation $m_{Ag}^\Omega[R]$ is used in place of $m_{Ag}^\Omega[\{R\}]$ for simplicity; likewise for the rest of this paper.

If S is not reliable, the information provided by S cannot be taken into account, and Ag's knowledge is vacuous:

$$m_{Ag}^\Omega[NR](\Omega) = 1. \tag{13}$$

Therefore, we have two non-vacuous pieces of evidence, $m_{Ag}^{\mathcal{R}}$ and $m_{Ag}^\Omega[R]$. Assuming that they are distinct, they can be combined by vacuously extending $m_{Ag}^{\mathcal{R}}$ to $\Omega\times\mathcal{R}$, computing the ballooning extension of $m_{Ag}^\Omega[R]$ in the same space, applying the CRC, and marginalizing the result on Ω:

$$m_{Ag}^\Omega[m_S^\Omega, m_{Ag}^{\mathcal{R}}] = \left(m_{Ag}^\Omega[R]^{\Uparrow\Omega\times\mathcal{R}} \circledcirc m_{Ag}^{\mathcal{R}\uparrow\Omega\times\mathcal{R}}\right)^{\downarrow\Omega}. \tag{14}$$

The resulting BBA $m_{Ag}^\Omega[m_S^\Omega, m_{Ag}^{\mathcal{R}}]$ (where the brackets [ ] indicate the evidential corpus, i.e., what is known by the belief holder Ag) only depends on m_S^Ω and α. Thus the discounted BBA $m_{Ag}^\Omega[m_S^\Omega, m_{Ag}^{\mathcal{R}}]$ of a BBA m_S^Ω is often denoted by ${}^\alpha m_S^\Omega$. It is equal to

$$\left\{\begin{array}{lcl} {}^\alpha m_S^\Omega(A) & = & (1-\alpha)m_S^\Omega(A), \quad \forall A\subset\Omega, \\ {}^\alpha m_S^\Omega(\Omega) & = & (1-\alpha)m_S^\Omega(\Omega)+\alpha, \end{array}\right. \tag{15}$$

which can be more simply written as:

$${}^\alpha m_S^\Omega = (1-\alpha)m_S^\Omega + \alpha\, VBF. \tag{16}$$

This operation was called discounting by Shafer [7, page 251], who introduced it on intuitive grounds. The formal justification presented here was proposed by Smets [9]. A generalization has been presented in [6].

2.3 De-Discounting

In this process, an agent Ag receives a BBA ${}^\alpha m_S^\Omega$ from a source S, different from the VBF and discounted with a discount rate $\alpha<1$. If Ag knows α, then it can recompute m_S^Ω by inversing the discounting operation (16):

$$m_{Ag}^\Omega = m_S^\Omega = \frac{{}^\alpha m_S^\Omega - \alpha\, VBF}{1-\alpha}, \tag{17}$$

that is,

$$\left\{\begin{array}{lcll} m_{Ag}^\Omega(A) & = & \frac{{}^\alpha m_S^\Omega(A)}{1-\alpha} & \forall A\subset\Omega, \\ m_{Ag}^\Omega(\Omega) & = & \frac{{}^\alpha m_S^\Omega(A)-\alpha}{1-\alpha} & . \end{array}\right. \tag{18}$$

This procedure is called *de-discounting* by Denœux and Smets in [2].

If α is not known, agent Ag can imagine all possible values in the range $[0, {}^\alpha m_S^\Omega(\Omega)]$. Indeed, as shown in [2], ${}^\alpha m_S^\Omega(\Omega)$ is the largest value for α such that the de-discounting operation (18) leads to a BBA. De-discounting ${}^\alpha m_S$ with this maximal value is called *maximal de-discounting.* The result is the *completely reinforced belief function* defined as follows.

Definition 7 (Completely reinforced BF) *Let m be a BBA different from the VBF. The* completely reinforced BBA ${}^{cr}m$ *associated with m is defined by:*

$${}^{cr}m(A) = \left\{\begin{array}{ll} \frac{m(A)}{1-m(\Omega)} & \forall A\subset\Omega, \\ 0 & \textit{otherwise} . \end{array}\right. \tag{19}$$

It corresponds to the BBA m such that the mass $m(\Omega)$ is totally and uniformly redistributed on focals elements of m. Denœux and Smets [2] has called this ${}^{cr}m$ the *maximal de-discounted BBA* as we explained.

Now that discounting and de-discounting are defined, it would be interesting to build some correcting mechanism allowing an agent to reinforce or weaken any pieces of information provided by a source. Zhu and Basir's extended scheme constitutes a first step in this direction.

2.4 Zhu and Basir's Extended Scheme

In [14], Zhu and Basir have proposed to extend the discounting process, in order to either augment or discount belief functions. This was achieved by retaining the discounting equation (16), and allowing the discount rate α to be in the range $[\frac{-m_S^\Omega(\Omega)}{1-m_S^\Omega(\Omega)}, 1]$:

- If $\alpha\in[0,1]$, ${}^\alpha m_S^\Omega$ is clearly a discounting of m_S^Ω;
- If $\alpha\in[\frac{-m_S^\Omega(\Omega)}{1-m_S^\Omega(\Omega)}, 0]$, ${}^\alpha m_S^\Omega$ is a de-discounting of m_S^Ω. Indeed, if α is replaced by $\frac{-\alpha'}{1-\alpha'}$ with $\alpha'\in$

$[0, m_S^\Omega(\Omega)]$, (16) is equivalent to the de-discounting equation (17):

$$\begin{array}{rcl} {}^{\alpha}m_S^\Omega & = & (1 - \frac{-\alpha'}{1-\alpha'}) m_S^\Omega + \frac{-\alpha'}{1-\alpha'}\ VBF \\ & = & \frac{m_S^\Omega - \alpha'\ VBF}{1-\alpha'}. \end{array} \quad (20)$$

This extended scheme has been successfully applied to a real-world application, but has not been formally justified. A justification is presented in Section 4 of this paper.

Before tackling general correction mechanisms, a general *reinforcement* scheme, for which the de-discounting process is shown to be a particular case, is firstly exposed.

3 De-discounting as a Particular Reinforcement Process

In this section, it is assumed that agent Ag receives a non-vacuous BBA m_S^Ω from a source S, describing the source's beliefs on the actual value of x. On the other hand, Ag knows that the source is reliable and has some knowledge on the *cautiousness* of S, quantified by a BBA $m_{Ag}^{\mathcal{C}}$ on the frame $\mathcal{C} = \{TC, NTC\}$, where TC stands for *"the source is too cautious"*, and NTC for *"the source is not too cautious"*. Let us assume that $m_{Ag}^{\mathcal{C}}$ has the following form:

$$\left\{ \begin{array}{rcl} m_{Ag}^{\mathcal{C}}(\{TC\}) & = & 1-\gamma \\ m_{Ag}^{\mathcal{C}}(\{NTC\}) & = & \gamma, \end{array} \right. \quad (21)$$

for some $\gamma \in [0, 1]$.

With these hypotheses, Agent Ag has two pieces of information: m_S^Ω and $m_{Ag}^{\mathcal{C}}$. To compute m_{Ag}^Ω, i.e., what finally agent Ag knows about the actual value of x, the notion of cautiousness has to be defined:

- If S is too cautious, the BBA provided by S should be completely reinforced:

$$m_{Ag}^\Omega[TC] = {}^{cr}m_S^\Omega \ . \quad (22)$$

- If S is not too cautious, the information provided by S become Ag's knowledge without reinforcement:

$$m_{Ag}^\Omega[NTC] = m_S^\Omega \ . \quad (23)$$

Therefore, $m_{Ag}^\Omega[m_S^\Omega, m_{Ag}^{\mathcal{C}}]$ depends only on three non-vacuous pieces of evidence: $m_{Ag}^{\mathcal{C}}$, $m_{Ag}^\Omega[TC]$ and $m_{Ag}^\Omega[NTC]$. Assuming that they are distinct, m_{Ag}^Ω can be computed using the following equation, similar to (14):

$$\begin{array}{rl} & m_{Ag}^\Omega[m_S^\Omega, m_{Ag}^{\mathcal{C}}] \\ = & \left(m_{Ag}^\Omega[TC]^{\Uparrow\Omega\times\mathcal{C}} \textcircled{\cap} m_{Ag}^\Omega[NTC]^{\Uparrow\Omega\times\mathcal{C}} \textcircled{\cap} m_{Ag}^{\mathcal{C}\uparrow\Omega\times\mathcal{C}} \right)^{\downarrow\Omega}. \end{array}$$

Proposition 1 *We have:*

$$m_{Ag}^\Omega = (1-\gamma)\ {}^{cr}m_S^\Omega + \gamma\ m_S^\Omega, \quad (24)$$

and, by replacing ${}^{cr}m_S^\Omega$ by its value:

$$\left\{ \begin{array}{rcl} m_{Ag}^\Omega(A) & = & (\frac{1-\gamma}{1-m_S^\Omega(\Omega)} + \gamma)\ m_S^\Omega(A) \quad \forall A \subset \Omega, \\ m_{Ag}^\Omega(\Omega) & = & \gamma\ m_S^\Omega(\Omega) \end{array} \right. \quad (25)$$

with $\gamma \in [0, 1]$.

Proof: See Appendix A.1.

We note the similarity between (24) and (16). Furthermore, by replacing $\gamma \in [0, 1]$ by $\frac{m_S^\Omega(\Omega)-\alpha}{m_S^\Omega(\Omega)(1-\alpha)}$ with $\alpha \in [0, m_S^\Omega(\Omega)]$, it can easily be checked that (25) is equivalent to de-discounting as defined by (18).

Actually, (24) defines a more *general reinforcement process* as the choice of ${}^{cr}m$ in (22) is somewhat arbitrary. Agent Ag could choose any more committed BBA than the original BBA given by the source. For instance, ${}^{cr}m$ in (22) could be replaced by the Baysesian BBA m_{bet} corresponding to the pignistic probability distribution [10]:

$$m_{bet}(\{\omega\}) = \sum_{\{A\subseteq\Omega, \omega\in A\}} \frac{m_S^\Omega(A)}{(1-m_S^\Omega(\emptyset))|A|},\ \forall\omega\in\Omega. \quad (26)$$

Thus, de-discounting process can be considered as a particular reinforcement process in which the completely reinforcement BBA is equal to the maximal de-discounted BBA. The quantity $1-\gamma$ indicates the amount of reinforcement, and the pourcentage of the mass on the universe which is distributed to focal elements.

In the next section, the definition of cautiousness is extended, leading to a more general correction mechanism and a justification of Zhu and Basir's scheme.

4 Justification of Zhu and Basir's Scheme

Let us adopt the same starting point as the previous section. Agent Ag receives a non-vacuous BBA m_S^Ω from a source S, and Ag has some knowledge about the *cautiousness* of S, quantified by a BBA $m_{Ag}^{\mathcal{C}}$ on the frame $\mathcal{C} = \{C, NC\}$. However, in this section, C stands for "the source is (very) cautious", and NC for "the source is not cautious at all", and the notion of cautiousness is now defined as follows:

- If S is very cautious, the information provided by S is maximally reinforced:

$$m_{Ag}^\Omega[C] = {}^{cr}m_S^\Omega. \quad (27)$$

- If S is not cautious at all, the information provided by S cannot be taken into account, and Ag's knowledge is vacuous:

$$m_{Ag}^\Omega[NC](\Omega) = 1. \quad (28)$$

Let us assume that $m_{Ag}^{\mathcal{C}}$ is defined, for some $\gamma \in [0, 1]$, by:

$$\left\{ \begin{array}{rcl} m_{Ag}^{\mathcal{C}}(\{C\}) & = & 1-\gamma \\ m_{Ag}^{\mathcal{C}}(\mathcal{C}) & = & \gamma\ . \end{array} \right. \quad (29)$$

Then, assuming that $m^{\mathcal{C}}_{Ag}$ and $m^{\Omega}_{Ag}[C]$ are distinct, m^{Ω}_{Ag} can be computed by:

$$m^{\Omega}_{Ag}[m^{\Omega}_S, m^{\mathcal{C}}_{Ag}] = \left(m^{\Omega}_{Ag}[C]^{\Uparrow\Omega\times\mathcal{C}} \circledcirc m^{\mathcal{C}\uparrow\Omega\times\mathcal{C}}_{Ag}\right)^{\downarrow\Omega}. \quad (30)$$

Proposition 2 *The resulting BBA m^{Ω}_{Ag} is equal to*

$$m^{\Omega}_{Ag} = (1-\gamma)\ {}^{cr}m^{\Omega}_S + \gamma\, VBF, \quad (31)$$

or, equivalently,

$$\left\{\begin{array}{lcll} m^{\Omega}_{Ag}(A) & = & (1-\gamma)\ {}^{cr}m^{\Omega}_S(A) & \forall A\subset\Omega, \\ m^{\Omega}_{Ag}(\Omega) & = & (1-\gamma)\ {}^{cr}m^{\Omega}_S(\Omega)+\gamma & \end{array}\right. \quad (32)$$

If ${}^{cr}m^{\Omega}_S$ is replaced by its value in (32):

$$\left\{\begin{array}{lcll} m^{\Omega}_{Ag}(A) & = & \frac{1-\gamma}{1-m^{\Omega}_S(\Omega)}\ m^{\Omega}_S(A) & \forall A\subset\Omega, \\ m^{\Omega}_{Ag}(\Omega) & = & \gamma & \end{array}\right. \quad (33)$$

with $\gamma\in[0,1]$.

Proof: m^{Ω}_{Ag} corresponds to a discounting of ${}^{cr}m^{\Omega}_S$, as C and NC plays the same role as R and NR in the discounting justification scheme, and m^{Ω}_S has been replaced by ${}^{cr}m^{\Omega}_S$ in (27).

□

If $\gamma = 0$, m^{Ω}_S is totally reinforced: $m^{\Omega}_{Ag}(A) = {}^{cr}m^{\Omega}_S(A)$, $\forall A\subseteq\Omega$.

If $\gamma = m^{\Omega}_S(\Omega)$, m^{Ω}_S remains unchanged: m^{Ω}_{Ag} is equivalent to m^{Ω}_S.

If $\gamma = 1$, m^{Ω}_S is totally discounted: $m^{\Omega}_{Ag}(\Omega)=1$.

It can be noticed that (33) is an other expression of Zhu and Basir's extended scheme recalled in Section 2.4. Indeed, by replacing $\gamma\in[0,1]$ by $\alpha(1-m^{\Omega}_S(\Omega)) + m^{\Omega}_S(\Omega)$ in (33), we get:

$$\left\{\begin{array}{lcl} m^{\Omega}_{Ag}(A) & = & (1-\alpha(1-m^{\Omega}_S(\Omega))-m^{\Omega}_S(\Omega))\frac{m^{\Omega}_S(A)}{1-m^{\Omega}_S(\Omega)} \\ & = & (1-\alpha)\ m^{\Omega}_S(A) \quad \forall A\subset\Omega, \\ & & \\ m^{\Omega}_{Ag}(\Omega) & = & \alpha(1-m^{\Omega}_S(\Omega))+m^{\Omega}_S(\Omega) \\ & = & \alpha-\alpha\ m^{\Omega}_S(\Omega)+m^{\Omega}_S(\Omega) \\ & = & (1-\alpha)m^{\Omega}_S(\Omega)+\alpha \end{array}\right.$$

with $\alpha\in[\frac{-m^{\Omega}_S(\Omega)}{1-m^{\Omega}_S(\Omega)},1]$, which is exactly Zhu and Basir's extended scheme.

As this operation is a discounting, coefficient γ, varying in $[0,1]$, can be automatically learnt from data using the expert tuning method introduced in [4] for instance. However, depending on $m^{\Omega}_S(\Omega)$, the same γ can lead to a discounting, or a de-discounting or a preservation of m^{Ω}_S, as illustrated by the following example.

Example 1 *Let us assume that the following BBAs come from the same sensor S and has been noted down at different times $i\in\{1,2,3\}$.*

$$\left\{\begin{array}{lcl} m_{S,1}(A) & = & 0.1 \\ m_{S,1}(B) & = & 0.1 \\ m_{S,1}(\Omega) & = & 0.8 \end{array}\right. \quad \left\{\begin{array}{lcl} m_{S,2}(A) & = & 0.2 \\ m_{S,2}(B) & = & 0.3 \\ m_{S,2}(\Omega) & = & 0.5 \end{array}\right.$$

$$\left\{\begin{array}{lcl} m_{S,3}(A) & = & 0.18 \\ m_{S,3}(B) & = & 0.72 \\ m_{S,3}(\Omega) & = & 0.10 \end{array}\right.$$

If $\gamma = 0.5$ is the rate used to correct this sensor, then:

$$\left\{\begin{array}{lclcl} m_{Ag,1}(A) & = & 5\,(1-\gamma)\ m_{S,1}(A) & = & 0.25 \\ m_{Ag,1}(B) & = & 5\,(1-\gamma)\ m_{S,1}(B) & = & 0.25 \\ m_{Ag,1}(\Omega) & = & \gamma & = & 0.50 \end{array}\right.$$

and, $m_{Ag,1}$ is a de-discounting of m_1.

$$\left\{\begin{array}{lclcl} m_{Ag,2}(A) & = & 2\,(1-\gamma)\ m_{S,3}(A) & = & 0.2 \\ m_{Ag,2}(B) & = & 2\,(1-\gamma)\ m_{S,3}(B) & = & 0.3 \\ m_{Ag,2}(\Omega) & = & \gamma & = & 0.5 \end{array}\right.$$

and, $m_{Ag,2}$ is equivalent to m_2.

$$\left\{\begin{array}{lclcl} m_{Ag,3}(A) & = & 10/9\,(1-\gamma)\ m_{S,3}(A) & = & 0.1 \\ m_{Ag,3}(B) & = & 10/9\,(1-\gamma)\ m_{S,3}(B) & = & 0.4 \\ m_{Ag,3}(\Omega) & = & \gamma & = & 0.5 \end{array}\right.$$

and, $m_{Ag,3}$ is a discounting of m_3.

The next section presents an other process of correction with more flexibilities, in particular with the control of the mass on the universe Ω.

5 A More General Correction Mechanism

In this section, a rational agent Ag receives a non-vacuous BBA m^{Ω}_S from a source S, and has some knowledge about the reliability and the cautiousness of S quantified by a BBA $m^{\mathcal{R}}_{Ag}$ on the space $\mathcal{R}$ composed of three states $\{R_1,R_2,R_3\}$.

If S is in state R_1, it means that S is totally unreliable, the information provided by S cannot be taken into account, and Ag's knowledge is vacuous:

$$m^{\Omega}_{Ag}[R_1](\Omega) = 1. \quad (34)$$

If S is in state R_2, then S is reliable and cautious enough, the information provided by S become Ag's knowledge:

$$m^{\Omega}_{Ag}[R_2] = m^{\Omega}_S. \quad (35)$$

If S is in state R_3, it means that S is too cautious, the information provided by S can be totally reinforced:

$$m^{\Omega}_{Ag}[R_3] = {}^{cr}m^{\Omega}_S. \quad (36)$$

Let us assume that $m^{\mathcal{R}}_{Ag}$ has the following form, with $\gamma_1+\gamma_2+\gamma_3=1$:

$$\left\{\begin{array}{lcl} m^{\mathcal{R}}_{Ag}(\{R_1\}) & = & \gamma_1, \\ m^{\mathcal{R}}_{Ag}(\{R_2\}) & = & \gamma_2, \\ m^{\mathcal{R}}_{Ag}(\{R_3\}) & = & \gamma_3. \end{array}\right. \quad (37)$$

Thus, the knowledge held by the agent Ag, knowing m^{Ω}_S and $m^{\mathcal{R}}_{Ag}$, can be computed from the three non-vacuous pieces of evidence, $m^{\mathcal{R}}_{Ag}$, $m^{\Omega}_{Ag}[R_2]$ and $m^{\Omega}_{Ag}[R_3]$, by the following formula:

$$m^{\Omega}_{Ag}[m^{\Omega}_S, m^{\mathcal{R}}_{Ag}] = \left(m^{\Omega}_{Ag}[R_2]^{\Uparrow\Omega\times\mathcal{R}} \circledcirc m^{\Omega}_{Ag}[R_3]^{\Uparrow\Omega\times\mathcal{R}} \circledcirc m^{\mathcal{R}\uparrow\Omega\times\mathcal{R}}_{Ag}\right)^{\downarrow\Omega}. \quad (38)$$

Proposition 3 *The resulting BBA m_{Ag}^{Ω} only depends on m_S^{Ω} and γ_i, $i \in \{1,2,3\}$. We have*

$$m_{Ag}^{\Omega} = \gamma_1 \, VBF + \gamma_2 m_S^{\Omega} + \gamma_3 {}^{cr}m_S^{\Omega}. \tag{39}$$

Equivalently,

$$\begin{cases} m_{Ag}^{\Omega}(A) &= (\gamma_2 + \frac{\gamma_3}{1-m_S^{\Omega}(\Omega)}) \; m_S^{\Omega}(A) \quad \forall A \subset \Omega, \\ m_{Ag}^{\Omega}(\Omega) &= \gamma_1 + \gamma_2 \; m_S^{\Omega}(\Omega) \; . \end{cases} \tag{40}$$

Proof: See Section A.2.

As $\gamma_1 + \gamma_2 + \gamma_3 = 1$:

- if $\gamma_1 = 0$, (40) is equivalent to (25), i.e., another expression of the de-discounting process (18);
- if $\gamma_2 = 0$, (40) is equivalent to (33), i.e., the Ω-controled correction;
- if $\gamma_3 = 0$, (40) is equivalent to the discounting (15).

With this correction process, the mass that remains on Ω is controled by γ_1. However, one more degree of freedom has been added as compared to the previous correction process.

Example 2 *Let us consider the same BBAs $m_{S,i}$ from the previous example, and suppose that $\gamma_1 = 0.1$, $\gamma_2 = 0.4$, $\gamma_3 = 0.5$ have been given by experts or learnt from data. Then:*

$$\begin{cases} m_{Ag,1}(A) &= (\gamma_2 + 5\;\gamma_3)\; m_{S,1}(A) &= 0.29 \\ m_{Ag,1}(B) &= (\gamma_2 + 5\;\gamma_3)\; m_{S,1}(B) &= 0.29 \\ m_{Ag,1}(\Omega) &= \gamma_1 + \gamma_2 \; m_{S,1}(\Omega) &= 0.52 \end{cases}$$

and, $m_{Ag,1}$ is a specialization of $m_{S,1}$, i.e., the mass provided by the source has been reinforced at this time.

$$\begin{cases} m_{Ag,2}(A) &= (\gamma_2 + 2\;\gamma_3)\; m_{S,2}(A) &= 0.28 \\ m_{Ag,2}(B) &= (\gamma_2 + 2\;\gamma_3)\; m_{S,2}(B) &= 0.42 \\ m_{Ag,2}(\Omega) &= \gamma_1 + \gamma_2 \; m_{S,2}(\Omega) &= 0.30 \end{cases}$$

and, at this time again, $m_{S,2}$ has been reinforced.

$$\begin{cases} m_{Ag,3}(A) &= (\gamma_2 + 10/9\;\gamma_3)\; m_{S,3}(A) &= 0.172 \\ m_{Ag,3}(B) &= (\gamma_2 + 10/9\;\gamma_3)\; m_{S,3}(B) &= 0.688 \\ m_{Ag,3}(\Omega) &= \gamma_1 + \gamma_2 \; m_{S,3}(\Omega) &= 0.140 \end{cases}$$

and, this time, the mass provided by the source has been adjusted by a weakening.

Remark 1 *If $m_{Ag}^{\mathcal{R}}$ has the following form:*

$$\begin{cases} m_{Ag}^{\mathcal{R}}(\{R_1\}) &= \gamma_1, \\ m_{Ag}^{\mathcal{R}}(\{R_2\}) &= \gamma_2, \\ m_{Ag}^{\mathcal{R}}(\{R_3\}) &= \gamma_3, \\ m_{Ag}^{\mathcal{R}}(\mathcal{R}) &= 1 - \gamma_1 - \gamma_2 - \gamma_3, \end{cases} \tag{41}$$

with $\gamma_1 + \gamma_2 + \gamma_3 \leq 1$, then m_{Ag}^{Ω} is equal to:

$$\begin{cases} m_{Ag}^{\Omega}(A) &= (\gamma_2 + \frac{\gamma_3}{1-m_S^{\Omega}(\Omega)}) \; m_S^{\Omega}(A) \qquad \forall A \subset \Omega, \\ m_{Ag}^{\Omega}(\Omega) &= 1 - \gamma_3 - \gamma_2(1 - m_S^{\Omega}(\Omega)) \; . \end{cases} \tag{42}$$

6 Conclusions

In this paper, some results have been presented concerning the de-discounting process, as well as several *correction mechanisms*, i.e., mechanisms for weakening or reinforcing belief functions.

A general *reinforcement* scheme has been firstly exposed. This scheme is based on the choice of a specialization operator, which is applied to the BBA provided by a source, when this source is considered to be too cautious. The de-discounting process is recovered where the specialization is chosen to be the completely reinforced BBA.

Concerning the general correction mechanisms, Zhu and Basir's extended scheme related to de-discounting and discounting processes has been justified: it corresponds to the discounting of the completely reinforced BBA, with a reparameterization. An even more general correction mechanism, with one additional degree of freedom, has also been defined.

As a perspective, the correction (reinforcement or weakening) performed by an agent may depend not only on the source, but also on the context, i.e., the true value of the variable of interest. Thus, in line with previous work by the authors on contextual discounting [6], a contextual correction mechanism could be designed and investigated.

A Proofs

A.1 Proof of Proposition 1

With the hypotheses of Section 3:

$$\begin{cases} m_{Ag}^{\mathcal{C}\uparrow\Omega\times\mathcal{C}}(\Omega \times \{TC\}) &= 1 - \gamma, \\ m_{Ag}^{\mathcal{C}\uparrow\Omega\times\mathcal{C}}(\Omega \times \{NTC\}) &= \gamma. \end{cases} \tag{43}$$

for all $A \subseteq \Omega$:

$$m_{Ag}^{\Omega}[TC]^{\Uparrow\Omega\times\mathcal{C}}(A \times \{TC\} \cup \Omega \times \{NTC\}) = {}^{cr}m_S^{\Omega}(A), \tag{44}$$

and, for all $B \subseteq \Omega$:

$$m_{Ag}^{\Omega}[NTC]^{\Uparrow\Omega\times\mathcal{C}}(B \times \{NTC\} \cup \Omega \times \{TC\}) = m_S^{\Omega}(B). \tag{45}$$

Moreover, for all $A \subseteq \Omega$ and $B \subseteq \Omega$:

$$\begin{aligned} & (A \times \{TC\} \cup \Omega \times \{NTC\}) \\ & \cap (B \times \{NTC\} \cup \Omega \times \{TC\}) \\ = \; & A \times \{TC\} \cup B \times \{NTC\}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} & (A \times \{TC\} \cup B \times \{NTC\}) \cap \Omega \times \{TC\} \\ = \; & A \times TC \;, \end{aligned} \tag{47}$$

$$\begin{aligned} & (A \times \{TC\} \cup B \times \{NTC\}) \cap \Omega \times \{NTC\} \\ = \; & B \times NTC \;. \end{aligned} \tag{48}$$

Thus, the combination $m_{Ag}^{\Omega}[TC]^{\Uparrow\Omega\times\mathcal{C}}$ ⓞ $m_{Ag}^{\Omega}[NTC]^{\Uparrow\Omega\times\mathcal{C}}$ ⓞ $m_{Ag}^{\mathcal{C}\uparrow\Omega\times\mathcal{C}}$, noted ⓞ$m_{Ag}^{\Omega\times\mathcal{C}}$, has two forms of focals elements, for all $A, B \subseteq \Omega$:

$$\begin{aligned} & ⓜm_{Ag}^{\Omega\times\mathcal{C}}(A\times\{TC\}) \\ = \ & (1-\gamma)\ {}^{cr}m_S^\Omega(A) \underbrace{\sum_{B\subseteq\Omega} m_S^\Omega(B)}_{=1}, \quad (49) \\ & ⓜm_{Ag}^{\Omega\times\mathcal{C}}(B\times\{NTC\}) \\ = \ & \gamma\ m_S^\Omega(B) \underbrace{\sum_{A\subseteq\Omega} {}^{cr}m_S^\Omega(A)}_{=1}. \quad (50) \end{aligned}$$

By simplifying, it remains two forms of focals elements, defined for all $A \subseteq \Omega$ by:

$$\begin{aligned} ⓜm_{Ag}^{\Omega\times\mathcal{C}}(A\times\{TC\}) &= (1-\gamma)\ {}^{cr}m_S^\Omega(A), \\ ⓜm_{Ag}^{\Omega\times\mathcal{C}}(A\times\{NTC\}) &= \gamma\ m_S^\Omega(A). \end{aligned}$$

Marginalizing this BBA on Ω finally gives:

$$m_{Ag}^\Omega(A) = (1-\gamma)\ {}^{cr}m_S^\Omega(A) + \gamma\ m_S^\Omega(A),\ \forall A\subseteq\Omega. \quad (51)$$

□

A.2 Proof of Proposition 3

With the hypotheses of Section 5:

$$\left\{ \begin{array}{lcl} m_{Ag}^{\mathcal{R}\uparrow\Omega\times\mathcal{R}}(\Omega\times\{R_1\}) &=& \gamma_1\ , \\ m_{Ag}^{\mathcal{R}\uparrow\Omega\times\mathcal{R}}(\Omega\times\{R_2\}) &=& \gamma_2\ , \\ m_{Ag}^{\mathcal{R}\uparrow\Omega\times\mathcal{R}}(\Omega\times\{R_3\}) &=& \gamma_3\ . \end{array} \right. \quad (52)$$

For all $A \subseteq \Omega$:

$$m_{Ag}^\Omega[R_2]^{\Uparrow\Omega\times\mathcal{R}}(A\times\{R_2\}\cup\Omega\times\{R_1,R_3\}) = m_S^\Omega(A). \quad (53)$$

For all $B \subseteq \Omega$:

$$m_{Ag}^\Omega[R_3]^{\Uparrow\Omega\times\mathcal{R}}(B\times\{R_3\}\cup\Omega\times\{R_1,R_2\}) = {}^{cr}m_S^\Omega(B). \quad (54)$$

Moreover, for all $A \subseteq \Omega$ and $B \subseteq \Omega$:

$$\begin{aligned} & (A\times\{R_2\}\cup\Omega\times\{R_1,R_3\}) \\ \cap\ & (B\times\{R_3\}\cup\Omega\times\{R_1,R_2\}) \\ =\ & A\times\{R_2\}\cup B\times\{R_3\}\cup\Omega\times\{R_1\}, \quad (55) \end{aligned}$$

and

$$\begin{aligned} & (A\times\{R_2\}\cup B\times\{R_3\}\cup\Omega\times\{R_1\})\cap(\Omega\times\{R_1\}) \\ & \qquad = \Omega\times\{R_1\}, \\ & (A\times\{R_2\}\cup B\times\{R_3\}\cup\Omega\times\{R_1\})\cap(\Omega\times\{R_2\}) \\ & \qquad = A\times\{R_2\}, \\ & (A\times\{R_2\}\cup B\times\{R_3\}\cup\Omega\times\{R_1\})\cap(\Omega\times\{R_3\}) \\ & \qquad = B\times\{R_3\}. \end{aligned}$$

Then the combination $m_{Ag}^\Omega[R_2]^{\Uparrow\Omega\times\mathcal{R}}$ ⓜ $m_{Ag}^\Omega[R_3]^{\Uparrow\Omega\times\mathcal{R}}$ ⓜ $m_{Ag}^{\mathcal{R}\uparrow\Omega\times\mathcal{R}}$, noted ⓜ$m_{Ag}^{\Omega\times\mathcal{R}}$, has three forms of focals elements, for all $A, B \subseteq \Omega$:

$$\begin{aligned} & ⓜm_{Ag}^{\Omega\times\mathcal{R}}(\Omega\times\{R_1\}) \\ =\ & \gamma_1 \sum_{A\subseteq\Omega} m_S^\Omega(A) \sum_{B\subseteq\Omega} {}^{cr}m_S^\Omega(B), \quad (56) \\ & ⓜm_{Ag}^{\Omega\times\mathcal{R}}(A\times\{R_2\}) \\ =\ & \gamma_2\ m_S^\Omega(A) \sum_{B\subseteq\Omega} {}^{cr}m_S^\Omega(B), \quad (57) \\ & ⓜm_{Ag}^{\Omega\times\mathcal{R}}(B\times\{R_3\}) \\ =\ & \gamma_3\ {}^{cr}m_S^\Omega(B) \sum_{A\subseteq\Omega} m_S^\Omega(A), \quad (58) \end{aligned}$$

or, equivalently, for all $A \subseteq \Omega$:

$$\begin{aligned} ⓜm_{Ag}^{\Omega\times\mathcal{R}}(\Omega\times\{R_1\}) &= \gamma_1, \\ ⓜm_{Ag}^{\Omega\times\mathcal{R}}(A\times\{R_2\}) &= \gamma_2\ m_S^\Omega(A), \\ ⓜm_{Ag}^{\Omega\times\mathcal{R}}(A\times\{R_3\}) &= \gamma_3\ {}^{cr}m_S^\Omega(A). \end{aligned}$$

Therefore, after marginalizing this BBA on Ω, one has:

$$\left\{ \begin{array}{lcl} m_{Ag}^\Omega(A) &=& \gamma_2\ m_S^\Omega(A) + \gamma_3\ {}^{cr}m_S^\Omega(A)\ \ \forall A\subset\Omega, \\ m_{Ag}^\Omega(\Omega) &=& \gamma_1+\gamma_2\ m_S^\Omega(\Omega)+\gamma_3\ {}^{cr}m_S^\Omega(\Omega)\quad \text{else}. \end{array} \right.$$

And, by replacing ${}^{cr}m_S^\Omega$ by its value, (40) is retrieved.

□

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